

Superfluidity and collective oscillations of trapped Bose-Einstein condensates in a periodical potential.

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Based on a unified theoretical treatment of the 1D Bogoliubov-de Gennes equations, the superfluidity phenomenon of the Bose-Einstein condensates (BEC) loaded into trapped optical lattice is studied. Within the perturbation regime, an all-analytical framework is presented enabling a straightforward phenomenological mapping of the collective excitation and oscillation character of a trapped BEC where the available experimental configurations also fit.

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INTRODUCTION

Harmonically trapped Bose-Einstein condensates (BECs) offer a great chance to understand the macroscopic quantum phenomena such as phase coherence [1–3] and matter wave diffraction [4]. Condensates loaded in a periodic potential forming an optical lattice (OL) [5, 6] may show a rich dynamic picture of oscillations as Bloch oscillations [7, 8], Belieav and Landau damping [9, 10], Landau-Zeeman tunneling [7], and the appearance of the superfluid oscillation of condensates [11, 12]. Although schemes of controlling the dynamics of BECs have been profusely described [8, 13–15], the underlying physics of some of these studies appears hidden under numerical analysis. The description in terms of excited states or Goldstone modes not only highlights the main cause of this behavior, but also enables the characterization of the BEC dynamics in universal terms. Thus, we select a platform of the BEC loaded simultaneously into a harmonic traps and an optical lattice to characterize the phenomenon of superfluidity as well as the dynamical properties and tackled the problem analytically. The purpose of this letter is to derive a perturbative treatment which allows explicit closed solutions for the phonon dispersion relation and to reveal the effect of BEC configurations on the superfluidity phenomenon. The method has undergone the test of comparison with experimental evidences with success.

Systems such as the cigar-shaped trap schematically represented in the upper panel of Fig. 1, can be considered quasi-one-dimensional (1D) confinement. Within the framework of mean field theory, the physical char-

acteristics of a BEC loaded in such trapping profile are ruled by the time dependent nonlinear Gross-Pitaevskii equation (GPE) [16].

$$i\hbar\partial_t|\Psi\rangle = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V_{ext} + \lambda_{1D}||\Psi||^2\right]|\Psi\rangle, \quad (1)$$

where λ_{1D} is the self-interaction parameter, m is the alkaline atom mass, $V_{ext}(x) = \frac{1}{2}m\omega_0^2x^2 - V_L \cos^2\left(\frac{2\pi}{d}x\right)$ represents the harmonic trap potential plus the periodic potential caused by the counter-propagating lasers with V_L the laser intensity, d its laser wavelength, and ω_0 the frequency of the harmonic trap. For Eq. (1), it is possible to prove rigorously, the existence of ground states for any λ_{1D} and V_L . Moreover, the set of ground states is orbitally stable and the ground states have a Gaussian-like exponential asymptotic behavior for any λ_{1D} , regardless of V_L value [17]. An important consequence of this mathematical fact is the stability of those physical magnitudes, such as no explosion and no damping as function of time, which are described by operators defined in the Hilbert space of the 1D GPE (1). This result is particularly related to the superfluidity properties, among others physical phenomena, of the harmonically confined condensates loaded in optical lattices.

The collective excitations, or so-called Goldstone modes of the BEC, can be obtained by applying a small deviation from the stationary solutions $|\Psi_0\rangle$ of Eq. (1),

$$|\Psi(t)\rangle = \exp(-i\mu t/\hbar)[|\Psi_0\rangle + |u\rangle \exp(-i\omega t) + |u\rangle \exp(-i\omega t) + |v^*\rangle \exp(i\omega t)], \quad (2)$$

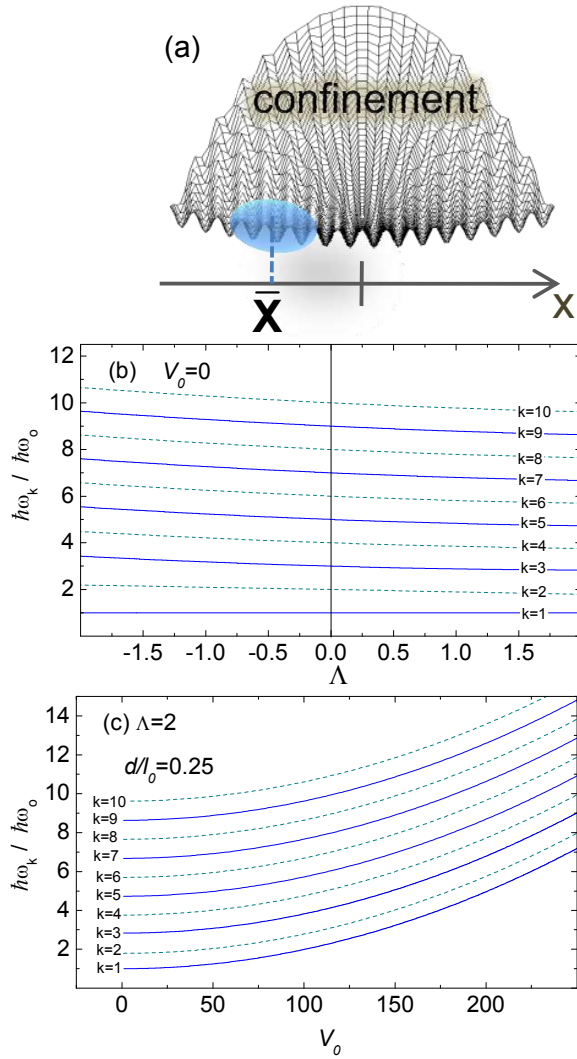


FIG. 1: (Color online). (a) The oscillating BEC loaded into a 1D optical lattice within a parabolic trap (grid). The elliptical spot symbolizes the condensate with a center of mass, $\bar{X}(t)$, oscillating around the trap bottom. (b) Collective excitation energies $\hbar\omega_k$ of the first 10 modes calculated as function of dimensionless parameters $\Lambda = \lambda_{1D}/(l_o\hbar\omega_0)$ for $V_o = 0$, and (c) as function of $V_o = V_L/\hbar\omega_0$ for $\Lambda = 2$ and $d/l_o = 0.25$. The solid (dashed) lines represent odd (even) modes.

which corresponds to linearizing the time-dependent nonlinear Schrödinger equation in terms of amplitudes $|u\rangle$ and $|v^*\rangle$, μ being the chemical potential and ω the mode or phonon frequencies [18]. Inserting $|\Psi(t)\rangle$ into Eq. (1), we obtain the Bogoliubov-de Gennes equations

$$\begin{bmatrix} \mathcal{L} & \lambda_{1D} |\Psi_0\rangle^2 \\ -\lambda_{1D} |\Psi_0\rangle^2 & -\mathcal{L} \end{bmatrix} \begin{bmatrix} |u\rangle \\ |v\rangle \end{bmatrix} = \hbar\omega_k \begin{bmatrix} |u\rangle \\ |v\rangle \end{bmatrix}, \quad (3)$$

where $\mathcal{L} = \hat{p}^2/2m + V_{ext} - \mu + 2\lambda_{1D} \langle \Psi_0 | \Psi_0 \rangle$. Three coupled nonlinear equations for $|\Psi_0\rangle$, $|u\rangle$ and $|v\rangle$ must be solved simultaneously, which incorporate the harmonic

trap and the stationary optical potential. In general, this is a very onerous task and it is not always possible to extract transparent solutions giving reliable information on the BEC dynamics. If the harmonic trap potential is switched off from Eqs. (1) and (3) it is possible analytically extract reliable information of the 1D condensate in a periodic potential as the Bloch oscillation and stability of the solution [19, 20]. This corresponds to the homogeneous case where the phonon wavevector $\mathbf{q} = q_x \mathbf{e}_x$ is a good quantum number and the excited frequency $\omega_{\mathbf{q}} = \omega(q)$ is a continuous function of \mathbf{q} . For $\omega_0 \neq 0$ the wavevector \mathbf{q} is no longer a good quantum number (inhomogeneous case) and the system (1 - 3) provides a set of discrete excited states ω_k labeled by $k = 1, 2, \dots$. By assuming a weakly-interacting Bose gas and not too strong laser intensities, the self-induced nonlinear interaction and the OL potential can be considered as perturbations with respect to the harmonic trap potential. Accordingly, compact solutions for μ and the spatial shape of the order parameter $|\Psi_0\rangle$, as determined by relevant parameters of the condensate are obtained [21].

NORMAL MODES

Considering the nonlinear term $\lambda_{1D} \langle \Psi_0 | \Psi_0 \rangle$ and the periodical potential $V_L \cos^2\left(\frac{2\pi}{d}x\right)$ as small terms compared to the confined harmonic trap potential strength, the solutions of the system (3) can be cast in terms of the complete set of harmonic oscillator wave functions $\{|\psi_n\rangle\}$ [22]: $|u\rangle = \sum_{n=0}^{\infty} A_n |\psi_n\rangle$ and $|v\rangle = \sum_{n=0}^{\infty} B_n |\psi_n\rangle$ and the coefficients A_n and B_n are expanded in form of series $A_n = A_n^{(1)} + A_n^{(2)} + \dots$; $B_n = B_n^{(1)} + B_n^{(2)} + \dots$, where the quantities $A_n^{(i)}$ and $B_n^{(i)} \sim \Lambda^i$, V_o^i . Using Eqs. (3) we get

$$\begin{aligned} & \sum_{n,i} A_n^{(i)} \left(B_n^{(i)} \right) \left[\left(k + \frac{1-V_o}{2} - \frac{\mu}{\hbar\omega_0} - (+) \frac{\omega_k}{\omega_0} \right) \delta_{k,n} \right. \\ & \quad \left. - \frac{1}{2} V_o \langle \psi_k | \cos 2\alpha x | \psi_n \rangle + 2\Lambda \langle \psi_k | \Psi_0 \rangle \langle \Psi_0 | \psi_n \rangle \right] \\ & = -\Lambda \sum_{n,i} B_n^{(i)} \left(A_n^{(i)} \right) \langle \psi_k | \Psi_0 \rangle \langle \Psi_0 | \psi_n \rangle, \end{aligned} \quad (4)$$

with $\alpha = 2\pi l_o/d$, $l_o = \sqrt{\hbar/m\omega_0}$, $\Lambda = \lambda_{1D}/(l_o\hbar\omega_0)$, $V_o = V_L/\hbar\omega_0$. Taking advantage of the procedure developed in Ref. 21 and solving simultaneously the system (4), it is possible to show that the independent phonon frequencies ω_k are given by

$$\begin{aligned} \frac{\omega_k}{\omega_0} = & k + \frac{\Lambda}{\sqrt{2\pi}} \left[-1 + \frac{2\Gamma(k+1/2)}{\sqrt{\pi}k!} \right] - \\ & \frac{V_o}{2} \exp(-\alpha^2) [L_k(2\alpha^2) - 1] - \\ & \frac{\Lambda V_o}{\sqrt{2\pi}} \exp(-\alpha^2) \left[Ei\left(\frac{\alpha^2}{2}\right) - \mathcal{C} - \ln \frac{\alpha^2}{2} + \frac{\delta_k(\alpha)}{\sqrt{\pi}} \right] + \\ & \frac{V_o^2}{4} \exp(-2\alpha^2) [Chi(2\alpha^2) - \mathcal{C} - \ln 2\alpha^2 + \rho_k(\alpha)] \\ & + \Lambda^2 \left[\frac{\gamma_k}{2\pi^2} + 0.033106 \right], \quad k = 1, 2, \dots, \end{aligned} \quad (5)$$

where $L_k(z)$, $\Gamma(z)$, $Ei(z)$, $Chi(z)$, and \mathcal{C} are the Laguerre polynomials, the gamma function, the exponential integral, the cosine hyperbolic integral, the Euler's constant, respectively. Finally, γ_k are numbers, δ_k and ρ_k being explicit functions the dimensionless parameter α [23, 24].

In Fig. 1 the analytical solutions for the frequencies ω_k are graphically represented for the first 10 modes. Panel (b) displays ω_k for attractive ($\Lambda < 0$) and repulsive ($\Lambda > 0$) cases at $V_0 = 0$. Also in panel (c) the influence of the periodical potential on the phonon modes being checked for fixed values $d/l_0 = 0.25$ and $\Lambda = 2$. In the first case, the collective oscillations show an almost flat dispersion as a function Λ . Note that the mode for $k = 1$ has the frequency value ω_0 of the harmonic trap [25]. In panel (c) is also seen a blue-shift renormalization of ω_k can be noted due to the presence of the OL.

The normalized eigenvectors $|\Phi_k\rangle^\dagger = (|u_k^*\rangle, |v_k\rangle)$ can be cast as

$$|\Phi_k\rangle = \begin{pmatrix} |\psi_k\rangle + \sum_{m \neq k} \frac{(4\Lambda f_{k,m} - V_o g_{k,m})}{2(k-m)} |\psi_m\rangle \\ - \sum_{m=0}^{\infty} \frac{\Lambda f_{k,m}}{k+m} |\psi_m\rangle \end{pmatrix}, \quad (6)$$

with $f_{k,m} = (-1)^{(k-m)/2} \Gamma\left(\frac{k+m+1}{2}\right) / (\pi\sqrt{2m!k!})$, $g_{k,m} = (-1)^{(k-m)/2} p! (2\alpha)^{|k-m|} L_p^{|k-m|}(2\alpha^2) 2^p / (\sqrt{2^{k+m} m! k!}) \times \exp(-\alpha^2)$, $L_p^t(z)$ being the Laguerre polynomials, $p = (k+m-|k-m|)/2$ and $m+k = \text{even number}$. Notice that the quasiparticle amplitudes $|u_k\rangle$ and $|v_k^*\rangle$ present parity inversion symmetry property, i.e. if the index k is an even or an odd number we are in presence of two independent subspaces where the wave functions $|\Phi_k\rangle$ become symmetric (even mode) or antisymmetric (odd mode) with respect the transformation $x \rightarrow -x$.

According to Eqs. (2) and (6), for a given time t , the probability density $|\langle x | \Psi_k(t) \rangle|^2$ of the excited states along the x -axis shows oscillations with well defined maxima which are quenched according to the exponential behavior $\exp(-x^2/l_o^2)$. The position of the maxima and minima of the axial density $|\langle x | \Psi_k(t) \rangle|^2$ are linked to the minima or maxima of the combined potential function $V/\hbar\omega_0 = 0.5(x/l_o)^2 - V_o \cos^2(2\pi x/l_o)$. Finally, another important limit is reached when the optical lattice

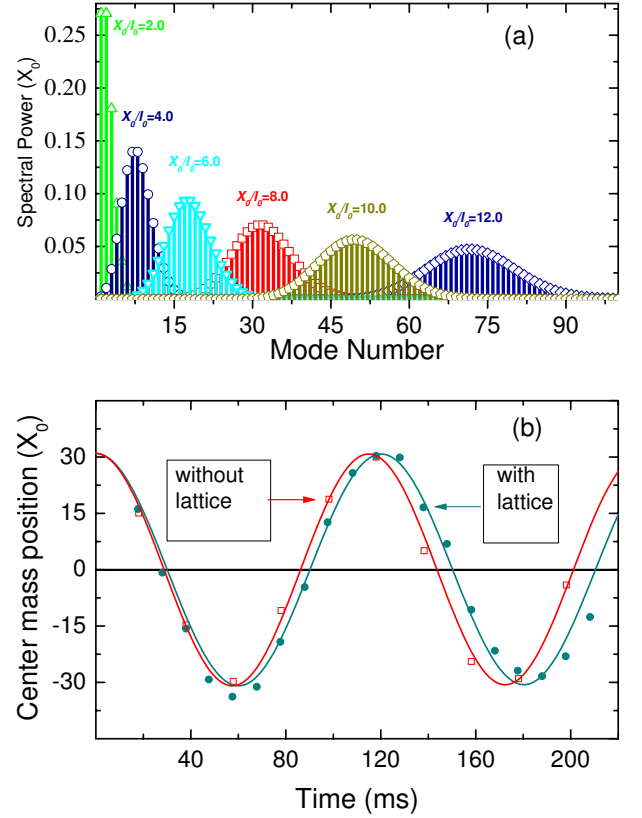


FIG. 2: (Color online) (a) Universal spectral power contributions to the BEC oscillations within the perturbation regime. The inset shows the mode number, k_m , where the maximum value of the spectral power is attained. (b) Center-of-mass position. Experimental data from Ref. 11 are represented by open squares and full circles, while by solid lines show the calculations after Eqs. (5) and (7).

is turned off. In this case the standard solutions (in a perturbative sense) of Bogoliubov-de Gennes equations for the inhomogeneous case are directly obtained from Eqs. (5) and (6) taking $V_o = 0$.

SUPERFLUIDITY.

To characterize the dynamics of a BEC loaded in an OL we evaluated the expectation value of the center-of-mass position $\bar{X}(t) = \langle \Psi(t) | x | \Psi(t) \rangle$, with $|\Psi(t)\rangle = \sum_{k=0}^{\infty} C_k |\phi_k(t)\rangle$. Here, the set of eigensolutions is chosen as $|\phi_0(t)\rangle = \exp(-i\mu t/\hbar) |\Psi_0\rangle$, $|\phi_k(t)\rangle = \exp(-i\mu t/\hbar) [\exp(-i\omega_k t) |u_k\rangle + \exp(i\omega_k t) |v_k^*\rangle]$ for $k \neq 0$, and the coefficients $\{C_k\}$ are obtained under certain initial condition. In our case, we consider that, at $t = 0$, the OL is absent and condensate is located out of equilibrium at certain distance X_0 from the origin, i.e. the order parameter $|\Psi_0\rangle$ is centered at $x = X_0$ with an ex-

pectation value $\overline{X(0)} = X_0$. At, $t > 0$ the system may or may not be loaded into the OL periodical potential.

A straightforward calculation, by keeping terms up to first-order in Λ and V_0 , yields that the dynamics of the center-of-mass is ruled by the equation $\overline{X(t)} = X_0(t) + X_{\Lambda, V_0}(t)$, where

$$X_0(t) = X_0 \exp \sum_{k=0}^{\infty} F_k \cos(\omega_{k+1} - \omega_k) t, \quad (7)$$

$F_k = \exp(-X_0^2/2l_o^2) \cdot X_0^{2k} / [(2l_o^2)^k k!]$ is the spectral power and $X_{\Lambda, V_0}(t)$ is a linear function of Λ and V_0 , with negligible contribution to $\overline{X(t)}$ for typical experimental setups. In Eq. 7 the condition $\omega_{k=0} = 0$ is used.

The spectral power contribution to these oscillations, in terms of the mode number, k , exclusively dependent on the relative initial position, X_0/l_0 , as displayed in Fig. 2 (a). For a given initial displacement, the mode $k = k_m$ with maximal contribution to the $X_0(t)$ is given by the equation $\ln(X_0^2/2l_o^2) - H_{k_m} - C = 0$, with $H_n = \sum_{k=1}^n 1/k$. Thus, one can see the larger the relative initial displacement, X_0/l_0 , the higher the main contributing modes, as shown in the inset of Fig. 2 (a), and wider their diffusion. This has energetic implications since the mode frequencies, ω_k , obtained from Eq. (5) are tunable with the nonlinear interaction strength and the OL parameters. Without OL the vibrational level spacing $\Delta\omega_k = \omega_{k+1} - \omega_k \approx \omega_0$ which make the expectation value $X_0(t) = X_0 \cos \omega_0 t$ [25].

Figure 2 (b) shows the superfluid oscillation for a BEC of ^{87}Rb extracted from Ref. 11 measured in a static magnetic trap with and without a 1D periodic potential, as simulated by our approach. Using Eqs. (5) and (7) we are able to reproduce the reported experimental center-of-mass position as function of time without using any fitting parameter. The oscillations observed in Fig. 2 (b) correspond to vibrational level spacing $\Delta\omega_k$ of the two independent subspaces with even and odd modes that make $X(t)$ oscillate with a frequency near the harmonic value ω_0 . The small frequency shift of the condensate loaded into OL, and observed in Fig. 2 (b), is directly linked to the renormalization of the atomic mass of the system moving in a periodical potential.

In summary, we presented a unified analytical description for the collective excitations (phonon frequencies ω_k , Eq. (5), the excited states wavefunctions $|\Phi_k\rangle$, Eq. (6)), the 1D superfluidity oscillation and the dynamics (center-

of-mass position $X_0(t)$ Eq. (7)) of BEC systems loaded in an optical lattice.

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- [1] Anderson B. P. and Kasevich M. A., *Science*, **282**, (1998) 1686.
 - [2] Hagley E. W. *et al.*, *Science*, **283**, (1999) 1706.
 - [3] Chiofalo M. L. and Tosi M. P., *Phys. Lett. A* **268**, (2000) 406.
 - [4] Ovchinnikov Y. B. *et al.*, *Phys. Rev. Lett.* **83**, (1999) 284.
 - [5] Jaksch D. *et al.*, *Phys. Rev. Lett.* **81**, (1998) 3108.
 - [6] Greiner M. *et al.*, *Nature (London)* **415**, (2002) 39.
 - [7] Raizen M. *et al.*, *Phys. Today* **50**, (1997) 30.
 - [8] Choi D.-I. and Niu Q., *Phys. Rev. Lett.* **82**, (1999) 2022.
 - [9] Katz N. *et al.*, *Phys. Rev. Lett.* **89**, (2002) 220401.
 - [10] Ferlaino F. *et al.*, *Phys. Rev. A* **66**, (2002) 011604.
 - [11] Burger S., *et al.*, *Phys. Rev. Lett.* **86**, (2001) 4447.
 - [12] Kagan Yu. and Maksimov L. A., *Phys. Rev. Lett.* **85**, (2000) 3075.
 - [13] Brezinova I., *et al.*, *Phys. Rev. A* **83**, (2011) 043611.
 - [14] Wanga Z., *et al.*, *J. Exp. Theor. Phys.* **112**, (2011) 355.
 - [15] Berry N. H. and Kutz J. N., *Phys. Rev. E*, **75** (2007) 036214.
 - [16] Gross E. P., *Nuovo Cimento* **20**, (1961) 454 ; Pitaevskii L. P., *Zh. Eksp. Teor. Fiz.* **40**, (1961) 646 [*Sov. Phys. JETP* **13**, (1961) 451].
 - [17] Cazenave T. and Lions P. L., *Commun. Math. Phys.* **85**, (1982) 549; R. Cipolatti, *et al.*, arXiv:1107.2704v1, (2011).
 - [18] Ruprecht P. A., *et al.*, *Phys. Rev. A*, **54**, (1996) 4178.
 - [19] Wu B. and Niu Q., *Phys. Rev. A*, **64**, (2001) 061603.
 - [20] Barontini G, and Modugno M. *Phys. Rev. A*, **76**, 041601 (2007)
 - [21] Trallero-Giner C., *et al.*, *Phys. Rev. A*, (2009) 063621.
 - [22] For a detailed description of the stationary solution $|\Psi_0\rangle$, the chemical potential and the validity of the perturbative method see Ref. 21.
 - [23] In typical experiments the values of the dimensionless parameter $\alpha = 2\pi l_o/d$ ranges between 10 and 50 (see Morsch O. and Oberthaler M., *Rev. Mod. Phys.* **78**, (2006) 179). This allows a simplification of the reported analytical expresions for the chemical potential μ and also for the excited frequencies ω_k .

[24]

$$\gamma_k = \frac{2}{\sqrt{\pi}} \sum_{m \neq 0} \frac{(-1)^{m+1} \Gamma^2(m + .5) \Gamma(k - m + .5)}{m m! k! 2^{2m}} - \frac{\Gamma^2(k + 0.5)}{2k(k!)^2} - \sum_{m \neq k} \left(\frac{1}{m+k} + \frac{4}{m-k} \right) \frac{\Gamma^2(\frac{m+1+k}{2})}{m! k!},$$

$$\delta_k(\alpha) = \frac{2}{\pi} \sum_{m \neq 0} \frac{(-2)^{m-1} \alpha^{2m} \Gamma^2(m + .5) \Gamma(k - m + .5)}{m (2m)! k!} + 2(-1)^k \sum_{m \neq k} \frac{(p)! (2\alpha)^{|m-k|} \Gamma(\frac{m+1+k}{2})}{(2)^{\frac{|m-k|}{2}} m! k! (k-m)} L_p^{|k-m|}(2\alpha^2),$$

and

$$\rho_k(\alpha) = \sum_{m \neq k} \frac{(p!)^2 (2\alpha^2)^{|m-k|}}{m! k! (k-m)} \left[L_p^{|k-m|}(2\alpha^2) \right]^2,$$

where \sum is restricted to $m + k = \text{even number}$, $p = (k +$

$m - |k - m|)/2$, and $L_p^t(z)$ are the Generalized Laguerre polynomials.

[25] Pitaevskii L. and Stringari S. , *Bose-Einstein Condensation*, (Clarendon Press, Oxford) 2003.